Infinite Limits

Consider a function like \( f(x) = \frac{1}{x^2}, x \neq 0 \). We find that numbers \( x \) close to 0 give values \( f(x) \) that are large and positive. For example

\[
\begin{align*}
  f(0.1) &= 100 & f(-0.04) &= 625 & f(0.0005) &= 4000000
\end{align*}
\]

In such a case, we say that \( f(x) \) approaches \( \infty \) as \( x \) approaches 0 and write \( \lim_{x \to 0} f(x) = \infty \). But do not interpret this to mean that there is a real number \( \infty \) with the property that \( |f(x) - \infty| \to 0 \) as \( x \to \infty \). These statements simply mean numbers \( x \) close to 0 give values \( f(x) \) that are large and positive.

Say we change the sign of \( \frac{1}{x^2} \) and let \( g(x) = -\frac{1}{x^2} \). We find that numbers \( x \) close to 0 give values \( g(x) \) that are large and negative. For that reason, we say that \( g(x) \) approaches \( -\infty \) as \( x \) approaches 0 and write \( \lim_{x \to 0} g(x) = -\infty \). Once again, we are not saying that there is a real number \( -\infty \) with the property that \( |f(x) - (-\infty)| \to 0 \) as \( x \) approaches 0. We write \( \lim_{x \to 0} g(x) = -\infty \) to convey the message that numbers \( x \) close to 0 give values \( g(x) \) that are large and negative.

We summarize these in an intuitive definition:

**Definition 1** Let \( f \) be a given function and \( c \) be a number which need not be in the domain of \( f \).

1. We say that \( f \) has limit \( \infty \) as \( x \) approaches \( c \), and write \( \lim_{x \to c} f(x) = \infty \), if numbers \( x \) close to \( c \) give values \( f(x) \) that are large and positive.

2. We say that \( f \) has limit \( -\infty \) as \( x \) approaches \( c \), and write \( \lim_{x \to c} f(x) = -\infty \), if numbers \( x \) close to \( c \) give values \( f(x) \) that are large and negative.

We can have infinite one-sided limits as the next example shows.

**Example 2** Let \( h(x) = \frac{x + 1}{x - 2}, x \neq 2 \). If \( x \) is close to 2 and is to the right of 2 then \( h(x) \) is a large positive number. For example,

\[
\begin{align*}
  h(2.005) &= 601, & h(2.001) &= 3001, & h(2.00002) &= 15001, & h(2.0000001) &= 30000001
\end{align*}
\]

Because of this, we say that \( h(x) \) approaches \( \infty \) as \( x \) approaches 2 from above, and write

\[
  h(x) \to \infty \text{ as } x \to 2^+ \quad \text{or} \quad \lim_{x \to 2^+} h(x) = \infty.
\]

On the other hand, if \( x \) is close to 2, and is to the left of 2 then \( h(x) \) is a large negative number. For example,

\[
\begin{align*}
  h(1.9905) &= -314.79, & h(1.995) &= -599, & h(1.9999998) &= -15000000
\end{align*}
\]

We say that \( h(x) \) approaches \( -\infty \) as \( x \) approaches 2 from below, and write

\[
  h(x) \to -\infty \text{ as } x \to 2^- \quad \text{or} \quad \lim_{x \to 2^-} h(x) = -\infty.
\]
Limit as $x$ Approaches $\infty$ or $-\infty$

We start with limits as $x$ approaches infinity. Only a function that is defined for all large positive numbers can have a limit as $x$ approaches $\infty$. Let $f$ be such a function. If every large positive number $x$ gives a value $f(x)$ close to a single number $l$, then we say that $f$ has limit $l$ as $x$ approaches $\infty$, and write $\lim_{x \to \infty} f(x) = l$.

Example 3 Consider $f(x) = \frac{1}{x^2}$. When $x$ is large and positive, then $x^2$ is a very large positive number, and so, its reciprocal is close to $0$. In other words, when $x$ is large and positive then $\frac{1}{x^2}$ is close to $0$, therefore $\lim_{x \to \infty} \frac{1}{x^2} = 0$.

Example 4 Let $g$ be defined by $g(x) = \frac{3x + 1}{x + 2}, x \neq -2$. Then $\lim_{x \to \infty} g(x) = 3$. To see this, note that when $x$ is large and positive, the dominant term in the numerator is $3x$ and the dominant term in the denominator is $x$, therefore $\frac{3x + 1}{x + 2}$ must be close to $\frac{3x}{x} = 3$. Another way of arriving at the same result is to divide the numerator and denominator of $\frac{3x + 1}{x + 2}$ by $x$, (the highest power of $x$ in the denominator). The result is $\frac{3 + \frac{1}{x}}{1 + \frac{2}{x}}$. When $x$ is large and positive, both $\frac{1}{x}$ and $\frac{2}{x}$ are small numbers close to $0$, hence $\frac{3 + \frac{1}{x}}{1 + \frac{2}{x}}$ should be close to $\frac{3 + 0}{1 + 0} = 3$. Therefore $\lim_{x \to \infty} \frac{3x + 1}{x + 2} = 3$.

Limit as $x$ Approaches $-\infty$

Only a function that is defined for all large negative numbers can have a limit as $x$ approaches $-\infty$. Let $f$ be such a function. If every large negative number $x$ gives a value $f(x)$ close to a single number $l$, then we say that $f$ has limit $l$ as $x$ approaches $-\infty$, and write $\lim_{x \to -\infty} f(x) = l$.

Example 5 Let $f(x) = 2^x$. Then $\lim_{x \to -\infty} f(x) = 0$.

Example 6 Let $f(x) = \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$. When $x$ is large and negative, the dominant term in the formula for $f$ is $2^{-x}$. Divide the numerator and denominator by $2^{-x}$ to get $f(x) = \frac{2^x - 1}{2^x + 1}$. Since $2^x$ is close to $0$ when $x$ is large and negative, it follows that $\lim_{x \to -\infty} f(x) = \frac{0 - 1}{0 + 1} = -1$.

Exercise 7

1. By definition, integer$(x)$ denotes the integer part of $x$. Thus integer$(5.79) = 5$ (simply throw away the decimal part), integer$(0.99) = 0$, integer$(-0.835) = 0$, integer$(-2.01) = -2$, etc. Consider the function $f(x) = \text{integer}(x)$.

   (a) Determine the following:

   i) $\lim_{x \to -2^+} f(x)$  ii) $\lim_{x \to -2^-} f(x)$  iii) $\lim_{x \to -5^+} f(x)$  iv) $\lim_{x \to -5^-} f(x)$

   (b) Give a formula for $\lim_{x \to n^+} f(x)$ and $\lim_{x \to n^-} f(x)$ when $n$ is a positive integer. Repeat when $n$ is a negative integer.

2. Let $g(x) = x \text{integer}(x)$.
(a) Determine the following:

\[ \lim_{x \to 0^+} g(x) \quad \lim_{x \to 0^-} f(x) \quad \lim_{x \to 0^+} g(x) \quad \lim_{x \to 0^-} g(x) \]

(b) Give a formula for \( \lim_{x \to n^+} g(x) \) and \( \lim_{x \to n^-} g(x) \) when \( n \) is an integer.

3. Draw the graph of \( h(x) = \begin{cases} 3 - 2x & \text{if } x < 1 \\ 4 + x & \text{if } x \geq 1 \end{cases} \) then determine the following limits:

\[ \lim_{x \to 1^-} h(x) \quad \lim_{x \to 1^+} h(x) \quad \lim_{x \to 1^+} h(x) \quad \lim_{x \to 1^-} h(x) \]

4. The expression \( |x| \) denotes the largest integer that is smaller than or equal to \( x \). For example,

\[ [6.21] = 6, \quad [0.92] = 0, \quad [-3] = -3, \quad [17] = 17, \quad [-0.2] = -1. \]

(a) Draw the graph of \( f(x) = |x| \) for values of \( x \) between \(-5\) and \(4\).

(b) Determine \( \lim_{x \to 2^+} f(x) \) and \( \lim_{x \to 2^-} f(x) \).

(c) Determine \( \lim_{x \to 3^+} f(x) \) and \( \lim_{x \to 3^-} f(x) \).

(d) Give a formula for \( \lim_{x \to n^+} f(x) \) and \( \lim_{x \to n^-} f(x) \) when \( n \) is an integer.

5. Let \( f(x) = |x| \), (the largest integer that is smaller than \( x \)), and \( g(x) = xf(x) \).

(a) Determine \( \lim_{x \to -1^+} g(x) \) and \( \lim_{x \to -1^-} g(x) \).

(b) Determine \( \lim_{x \to n^+} g(x) \) and \( \lim_{x \to n^-} g(x) \) where \( n \) is an integer.

(c) Determine \( \lim_{x \to a^+} g(x) \) and \( \lim_{x \to a^-} g(x) \) where \( a \) is a real number which is not an integer.

6. Let \( f(x) = \frac{x^2 + 1}{x + 1}, x \neq -1 \). Determine \( \lim_{x \to -1^+} f(x) \) and \( \lim_{x \to -1^-} f(x) \).

7. Let \( g(x) = \frac{1}{x^2}, x \neq 0 \). Determine \( \lim_{x \to 0^+} g(x) \) and \( \lim_{x \to 0^-} g(x) \).

8. Let \( f(x) = \frac{3x - 3 - x}{3x + 3 - x} \). Determine \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

9. Neural tissue can be electrically excited if the current across the cell membrane exceeds the threshold current. The threshold current \( I \) is related to the duration of time \( t \) that current flows across the membrane by the equation

\[ I(t) = \frac{a}{t} + b, \ t > 0 \]

where \( a \) and \( b \) are positive constants. Sketch the graph of \( I \) then determine and give the physical interpretation of: (i) \( \lim_{x \to 0^+} I(t) \) and (ii) \( \lim_{x \to \infty} I(t) \).

10. **A precise definition of an infinite limit**: Say we have to show that \( f(x) = \frac{1}{(x - 2)^2} \) has limit \( \infty \) as \( x \) approaches \( 2 \). Then we have to convince every individual that \( f(x) \) is large and positive when \( x \) is close to \( 2 \). The first question we have to address is: "what is a large positive number?" Unfortunately, the answer is: it depends on who you ask! That being the case we have to be flexible. We have to let an individual choose what he/she considers to be a large positive number. Let it be \( K \). Then to convince him or her that numbers close to \( 2 \) give values that are large and positive, we simply have to produce a punctured interval \((c - \delta, c) \cup (c, c + \delta) \) with the property that every number \( x \) in the interval gives a value \( f(x) \) that is bigger than \( K \). To be in a position of convincing everybody that shows up, we should be able to do this for every positive number \( K \). This suggests the following definition:
A function $f(x)$ has limit $\infty$ as $x$ approaches a number $c$ if, given any positive number $K$, we can find a positive number $\delta$ such that $f(x) > K$ for all $x$ in $(c-\delta, c) \cup (c, c+\delta)$.

Alternatively:

A function $f(x)$ has limit $\infty$ as $x$ approaches a number $c$ if, given any positive number $K$, we can find a positive number $\delta$ such that $f(x) > K$ if $0 < |x-c| < \delta$.

In the case of $f(x) = \frac{1}{(x-2)^2}$, let $K > 0$ be given. We have to find a positive number $\delta$ such that $\frac{1}{(x-2)^2} > K$ if $0 < |x-2| < \delta$. This inequality is satisfied by any $x$ such that $|x-2| < \frac{1}{\sqrt{K}}$.

It follows that if we take any $\delta < \frac{1}{\sqrt{K}}$ then $\left| \frac{1}{(x-2)^2} \right| > K$ if $|x-2| < \delta$.

11. Complete the following precise definition of an infinite limit as $x$ approaches a number from above:

A function $f$ has limit $1$ as $x$ approaches a number $c$ from above if given any positive number $K$, we can find a positive number $\delta$ such that ... 

Use the above precise definition to show that $f(x) = \frac{1}{\sqrt{x}}$ has limit $\infty$ as $x$ approaches $0$ from above.

12. A precise definition of a limit as $x$ approaches $\infty$ or $-\infty$: As we pointed out, a function $f(x)$ has limit $\infty$ as $x$ approaches $\infty$ if every number $x$ that is large and positive gives a value $f(x)$ that is close to $0$. Using the idea of a large positive number developed in Exercise 10 above, a precise way of saying this is the following:

A function $f(x)$ has limit $l$ as $x$ approaches $\infty$ if given any positive number $\varepsilon$ we can find a positive number $K$ such that $|f(x) - l| < \varepsilon$ when $x > K$.

A function $f(x)$ has limit $m$ as $x$ approaches $-\infty$ if given any positive number $\varepsilon$ we can find a positive number $N$ such that $|f(x) - m| < \varepsilon$ when $x < -N$.

For an example, we show that $f(x) = \frac{2x}{x-3} + 1$ has limit $2$ as $x$ approaches $\infty$. To this end, let $\varepsilon > 0$. We have to find a positive number $K$ such that $\left| \frac{2x}{x-3} - 2 \right| < \varepsilon$ when $x > K$. We simplify the inequality to get

$$\left| \frac{6}{x-3} \right| < \varepsilon$$

This is satisfied if $|x-3| > \frac{6}{\varepsilon}$. We may assume that $x > 3$. Then $x - 3 > \frac{6}{\varepsilon}$, which simplifies to $x > 3 + \frac{6}{\varepsilon}$. Therefore if we take any $K > 3 + \frac{6}{\varepsilon}$ then $\left| \frac{2x}{x-3} - 2 \right| < \varepsilon$ when $x > K$.

(a) Use the precise definition of a limit as $x$ approaches $\infty$ to show that $f(x) = \frac{x}{3x+1}$ has limit $\frac{1}{3}$ as $x$ approaches $\infty$.

(a) Use the precise definition of a limit to show that $g(x) = \frac{4x-1}{x^2} + 1$ has limit $4$ as $x$ approaches $-\infty$. 

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