Factoring some Trinomials by Grouping

We will apply this technique to quadratic expressions. The following examples illustrate the steps:

Example 1 *To factor $x^2+3x+2x+6$ by grouping, we note that the terms x^2 and $3x$ have a common factor, namely x. Also the terms $2x$ and 6 have a common factor, namely 2. Therefore we put parenthesis around x^2+3x. We do the same around $2x+6$. The result is*

$$x^2+3x+2x+6=\left(x^2+3x\right)+(2x+6)$$

The next step is to pull out the greatest common factors from each group to get

$$x^2+3x+2x+6=x\,(x+3)+2\,(x+3)$$

Note that we now have a new common factor, namely $(x+3)$. When we pull it out we get

$$x^2+3x+2x+6=(x+3)\,(x+2)$$

In practice the expression $x^2+3x+2x+6$ of Example 1 would be given as a trinomial x^2+5x+6, (it is called a trinomial because it has three terms). Then you have to figure out how to split the middle term $5x$ to get an expression that can be factored by grouping. A clue comes from expanding $(x+a)\,(x+b)$. The result is

$$\begin{aligned}(x+a)\,(x+b) &= x^2+xb+ax+ab\\ &= x^2+(b+a)\,x+ab\end{aligned}$$

We now see that the coefficient $(b+a)$ of the middle term $(b+a)\,x$ is the sum of two numbers whose product is last term ab. Applying this to x^2+5x+6, we conclude that to factor x^2+5x+6 by grouping, we must look for two numbers whose product is 6, (the last term), and whose sum is 5, (the coefficient of x in the middle term $5x$). These numbers must have the same sign, else their product would not be positive. Since their sum must be 5 which is also positive, both numbers must be positive. The positive pairs of integers whose product is 6 are 1,6 and 2,3. It is the second pair that satisfies the the additional condition that their sum is 5. Therefore we write

$$x^2+5x+6=x^2+2x+3x+6$$

Actually, the order is not important, so we may also write it as

$$x^2+5x+6=x^2+3x+2x+6$$

If we write it as $x^2+5x+6=x^2+2x+3x+6$ then grouping gives

$$\begin{aligned}x^2+5x+6 &= x^2+2x+3x+6=\left(x^2+2x\right)+(3x+6)\\ &= x\,(x+2)+3\,(x+2)=(x+2)\,(x+3)\end{aligned}$$

We have already handled the case $x^2+5x+6=x^2+3x+2x+6$ in Example 1, and as you can see, we get the same result.

Example 2 *To factor x^2+6x+8 by grouping, we look for two numbers whose product is 8 and have whose sum 6. The factors must have the same sign, (because their product must be positive). Since their sum must also be positive, they must be positive. There are two pairs of positive integers whose product is 8 and they are 1,8 and 2,4. It is the pair 2, 4 which satisfies the additional condition that their sum is 6. Therefore*

$$\begin{aligned}x^2+6x+8 &= x^2+2x+4x+8=\left(x^2+2x\right)+(4x+8)\\ &= x\,(x+2)+4\,(x+2)=(x+2)\,(x+4)\end{aligned}$$

Example 3 *To factor $x^2 - 2x - 15$ by grouping, we look for two numbers with product -15 and sum -2. The factors must have opposite signs, because that their product has to be negative. There are four pairs of integers with product -15 and they are (i) $-15, 1$ (ii) $-1, 15$ (iii) $-5, 3$ and $5, -3$. Of these, it is the pair $-5, 3$ which satisfies the additional condition that their sum is -2. Therefore we write*

$$\begin{aligned} x^2 - 2x - 15 &= x^2 + 3x - 5x - 15 = \left(x^2 + 3x\right) + (-5x - 15) \\ &= x(x+3) - 5(x+3) = (x+3)(x-5) \end{aligned}$$

Example 4 *To factor $x^2 - 7x + 12$ by grouping, we look for two numbers with product 12 and sum -7. The numbers must have the same sign in order for their product to be positive. Since their sum must be -7 which is negative, both numbers must be negative. They are -3 and -4. Therefore*

$$\begin{aligned} x^2 - 7x + 12 &= x^2 - 3x - 4x + 12 = \left(x^2 - 3x\right) - (4x - 12) \\ &= x(x-3) - 4(x-3) = (x-3)(x-4) \end{aligned}$$

Example 5 *A special case: To factor $x^2 + 2ax + a^2$ where a may be any number: We look for two numbers with product a^2 and sum $2a$. They are a and a, therefore*

$$\begin{aligned} x^2 + 2ax + a^2 &= x^2 + ax + ax + a^2 = \left(x^2 + ax\right) + \left(ax + a^2\right) \\ &= x(x+a) + a(x+a) = (x+a)(x+a) = (x+a)^2 \end{aligned}$$

In particular,

$$x^2 + 6x + 9 = x^2 + 2(3)x + (3)^2 = (x+3)^2,$$

$$x^2 + 14x + 49 = x^2 + 2(7)x + (7)^2 = (x+7)^2$$

Example 6 *Another special case: To factor $x^2 - 2bx + b^2$ where b may be any number: We look for two numbers whose product is b^2 and whose sum is $-2b$. They are $-b$ and $-b$, therefore*

$$\begin{aligned} x^2 - 2bx + b^2 &= x^2 - bx - bx + a^2 = \left(x^2 - bx\right) - \left(bx - b^2\right) \\ &= x(x-b) - b(x-a) = (x-b)(x-b) = (x-b)^2 \end{aligned}$$

In particular,

$$x^2 - 18x + 81 = x^2 - 2(9)x + (9)^2 = (x-9)^2,$$

$$x^2 - 10x + 25 = x^2 - 2(5)x + (5)^2 = (x-5)^2$$

Exercise 7 *Factor each algebraic expression:*

(a) $x^2 - 2x - 15$	*(b)* $x^2 - 8x + 7$	*(c)* $x^2 + 2x - 8$
(d) $x^2 + 8x + 12$	*(e)* $x^2 + 9x + 8$	*(f)* $x^2 - 2x - 8$
(g) $x^2 - 11x + 30$	*(h)* $x^2 - 6x + 9$	*(i)* $x^2 + 10x + 16$
(j) $x^2 + 12x + 36$	*(k)* $x^2 - 2x + 1$	*(l)* $x^2 + 4x - 5$

Factoring More General Trinomials

We now address more general trinomials than the special ones we encountered in the previous section. A general one has the form $ax^2 + bx + c$ where a, b and c are constants. Examples: (i) $3x^2 + 4x + 1$, (ii) $6x^2 - 5x - 6$ and (iii) $4x^2 + 7x - 2$. In each case, the coefficient of x^2 is NOT 1.

Take $4x^2 + 7x - 2$ for illustration. Multiply 4, (the coefficient of x^2), by the constant term -2, to get -8. Now look for two numbers whose product is -8 and whose sum is 7, (the coefficient of x). They are 8 and -1. Split $7x$ as $8x - x$ and proceed as before:

$$4x^2 + 7x - 2 = 4x^2 + 8x - x - 2 = \left(4x^2 + 8x\right) - (x+2) = 4x(x+2) - (x+2) = (x+2)(4x-1)$$

The reason for this step is that we must find terms $(px+r)$ and $(qx+s)$ such that

$$(px+r)(qx+s)=4x^2+7x-2$$

When we remove parentheses in $(px+r)(qx+s)$, we get

$$4x^2+7x-2=(pq)x^2+(ps+rq)x+rs. \tag{1}$$

Now we wee that the coefficient of x, which is $7=ps+rq$ is the sum of two factors ps and rq of $psrq=4\times(-2)$. Here are two more examples:

Example 8 *To factor $6x^2-5x-6$, we look for two numbers that have product $(6)(-6)=-36$ and sum -5. They are -9 and 4. Therefore we split up $-5x$ as $-9x+4x$ then proceed to factor by grouping:*

$$6x^2-5x-6=6x^2-9x+4x-6=\left(6x^2-9x\right)+(4x-6)=3x(2x-3)+2(2x-3)=(2x-3)(3x+2)$$

Example 9 *To factor $12x^2-7x-12$:*

> *The product of 12 and -12 is -144. Therefore we need two numbers whose product is -144 and sum is -7. Since their product must be negative, one of them must be negative and the either one positive. If we write -144 as*
>
> $$(-12)(12)=(-4)(3)(4)(3)$$
>
> *we see that $(-4)(4)=-16$, $(3)(3)=9$ and the sum of these two numbers is -7. Therefore we should split $-7x$ as $-16x+9x$ and the result is*
>
> $$\begin{aligned} 12x^2-7x-12 &= \left(12x^2-16x\right)+(9x-12)=4x(3x-4)+3(3x-4) \\ &= (4x+3)(3x-4) \end{aligned}$$

Exercise 10

1. *Factor the following trinomials*

(a) $2x^2-x-6$	*(b)* $12x^2+5x-3$	*(c)* $2x^2+13x+15$
(d) $12x^2-19x+4$	*(e)* $9x^2-9x-4$	*(f)* $8x^2+6x+1$
(g) $16x^2+16x+3$	*(h)* $4x^2+5x-6$	*(i)* $10x^2-17x+3$
(j) $5x^2-6x+1$	*(k)* $6x^2-13x+6$	*(l)* $10x^2+13x-3$

Example 11 *We usually have to factor quadratic expressions in order to solve quadratic equations. For example, to solve the equation $3x^2-11x-4=0$, we first factor the left hand side. The result is*

$$(3x+1)(x-4)=0$$

Now we argue that either $3x+1=0$ or $x-4=0$ (since the product of two non-zero numbers cannot be zero). Therefore $3x=-1$ or $x=4$. The solutions are $x=-\frac{1}{3}$ or $x=4$.